Journal of Statistical Physics, Vol. 65, Nos. 1/2, 1991

Basins of Periodic Orbits for Elliptic Maps of the Torus

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Received April 4, 1991

The behavior of basins of periodic orbits, for families of elliptic maps in the 2D torus depending on a parameter, is studied. We give an explicit formula for periodic orbits (i.e., central points of basins), considering also the occurrence of singular situations. Such a formula describes the evolution of basins, showing that onset and disappearance of periodic orbits cannot be reduced to a simple bifurcation scheme. Also, the stochastic features of the strange attractor at the border of ellipticity may be related to the dynamics of collapsing basins.

KEY WORDS: Elliptic maps; basins; periodic orbits.

The onset of stochasticity in dissipative dynamical systems is strictly involved with the behavior of the basins of attraction of periodic orbits, as shown, for instance, by the class of systems studied within the Feigenbaum scheme of "period doubling."[1,2] As far as we know, analogous general results do not exist for two-dimensional mappings.

Consider, for instance, the following family of dissipative automorphisms in the two-dimensional torus:

$$T_\theta = \begin{pmatrix} 2\cos\theta & \cos\theta - \sin\theta \\ \cos\theta + \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

which has been studied in ref. 3 for $0 \leqslant \theta \leqslant \pi/2$. At the limit values, this family reduces to two remarkable cases of conservative systems: $T_{\pi/2}$ is the well known Anosov "cat" automorphism, a prototype for chaotic dynamical systems; on the contrary, T_0, a simple rotation of period 4 for all the points in the torus, may be seen as a discrete analog of an integrable system. The

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0022-4715/91/1000-0363$06.50/0

continuous dependence on the parameter θ and the opposite statistical features at the extremes imply some type of order–chaos transition. In ref. 3, mostly by a numerical approach, it was shown indeed that: (a) the onset of chaos settles down at $\theta_0 = \arcos(2/3)$, when the automorphism becomes hyperbolic; and (b) such a transition is strictly related to the collapse of basins around periodic orbits, with the appearance of a strange attractor. The numerical approach, however, did not clarify the features of the collapse, even because of the very irregular behavior of the basins.

Systems like (1) are an example of the larger class of discontinuous one-parameter elliptic automorphisms of the two-dimensional torus that we shall consider here. The relevance of these systems, besides the fact that they could model some very peculiar return maps of flows, consists in displaying an unusual route to chaos in two dimensions. Moreover, with a direct and analytical insight into the behavior of the basins, we shall obtain as a by-product an explicit algorithm to find periodic orbits and a method to evaluate their existence intervals. Finally, such an investigation will indicate why a simple "universal" behavior cannot be expected in these cases, and a link between the basin dynamics and the features of the chaotic attractor at the border of ellipticity domain.

Introducing the notation, we recall some facts about discrete elliptic systems in $\mathbf{R}^2$. Let T be a unitary ellitic 2×2 matrix, i.e., a matrix with eigenvalues $\lambda_{\pm} = e^{\pm i\phi}$, where $\phi = \arcos[\mathrm{Tr}(T/2)]$. $\mathbf{R}^2$ is foliated by T-invariant parallel ellipses centered in the origin.[4] This means that orbits starting from every $\mathbf{z}_0 \in \mathbf{R}^2$ (i.e., sequences $\{\mathbf{z}_0, T\mathbf{z}_0, T^2\mathbf{z}_0,...\}$) either fill densely the ellipse passing through $\mathbf{z}_0$, or are finitely periodic on it. Precisely, if $\phi/2\pi = m/q$ is rational, the orbit is q-periodic; otherwise, it is

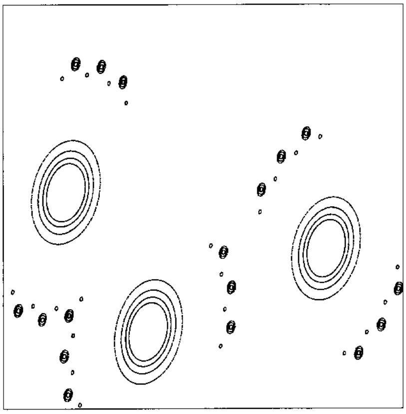

Fig. 1. For $\theta = 1.1988$, four orbits in a period-3 basin, three in a period-17 basin, and one in a period-23 basin.

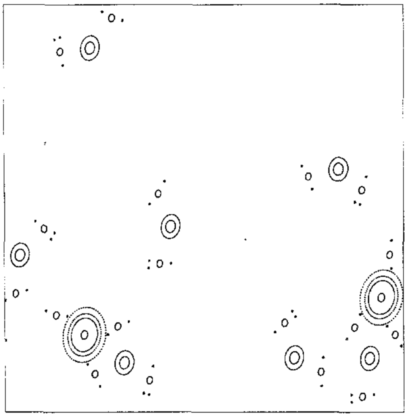

Fig. 2. For $\theta = 1.35$, four orbits in a period-2 basin, two in a period-7 basin, one in a period-18 basin, and one in a period-44 basin.

dense. There exists indeed a transformation S of $\mathbf{R}^2$ which maps ellipses into circles centered in the origin, the motion on them, STS^{-1}, being a rotation of an angle ϕ.

Let $\mathscr{T}^2$ be the 2D torus represented by the unit square with identification of the opposite sides. We define $\tilde{T}$ as the nonlinear system induced by T in $\mathscr{T}^2$. $\tilde{T}$ results to be dissipative, in general, because pieces of $\mathbf{R}^2$ may overlap in $\mathscr{T}^2$. A direct inspectation shows that, possibly after a transient, orbits in $\mathscr{T}^2$ lie in "basins" centered around periodic orbits [as shown, e.g., in Figs. 1–3, referring, like the following ones, to the family (1) of automorphisms T_θ].

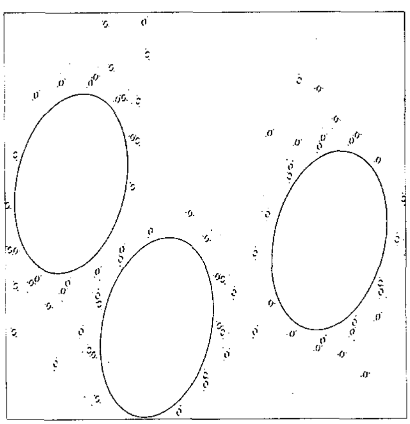

Fig. 3. For $\theta = 1.1799$, one single orbit in a period-3 basin after a transient near a period-75 basin.

Note that the explicit form of $\tilde{T}$ depends on the point it applies to: to every orbit $\{\tilde{\mathbf{z}}_k\}$ is associated a sequence of integer vectors $\{\mathbf{n}_k\}$ defined by

$$\tilde{\mathbf{z}}_{k+1} \equiv \tilde{T}\tilde{\mathbf{z}}_k = T\tilde{\mathbf{z}}_k - \mathbf{n}_k \tag{2}$$

and the form of $\tilde{T}$ is completely defined by $\{\mathbf{n}_k\}$. Of course, if $\mathscr{C}_p \equiv \{\tilde{\mathbf{z}}_1,..., \tilde{\mathbf{z}}_p\}$ is a p-periodic orbit in $\mathscr{T}^2$ (i.e., $\tilde{T}^p\tilde{\mathbf{z}}_k = \tilde{\mathbf{z}}_{k+p} = \tilde{\mathbf{z}}_k$), then the associated sequence is also p-periodic, and it may be called the "characteristic sequence" of $\mathscr{C}_p$.

Lemma. For a p-periodic orbit $\mathscr{C}_p$, let $\{\mathbf{n}_k\}$ be the associated characteristic sequence defined by (2), and $\mathscr{E}_0$ a T-invariant ellipse (i.e., $T\mathscr{E}_0 = \mathscr{E}_0$). If, for $k = 1,..., p$, $\mathscr{E}_0$ is so small that

$$\tilde{\mathbf{z}}_k + \mathscr{E}_0 \equiv \mathscr{E}_k \subset \mathscr{T}^2 \tag{3}$$

(where $\tilde{\mathbf{z}}_k + \mathscr{E}_0$ are the points $\tilde{\mathbf{z}}_k + \tilde{\mathbf{z}}$, $\tilde{\mathbf{z}} \in \mathscr{E}_0$), then

$$\tilde{T}\mathscr{E}_0 = T\mathscr{E}_{\mathrm{k}} - \mathbf{n}_k = \mathscr{E}_{k+1}$$

In other words, the set of p ellipses defined by (3) is invariant for $\tilde{T}$, and every single $\mathscr{E}_k$ is invariant for $\tilde{T}^p$.

The proof follows immediately from relation (2):

$$T\mathscr{E}_k - \mathbf{n}_k = T(\tilde{\mathbf{z}}_k + \mathscr{E}_0) - \mathbf{n}_k = \tilde{\mathbf{z}}_{k+1} + \mathscr{E}_0 \equiv \mathscr{E}_{k+1}$$

Note 1. When condition (3) holds for $\mathscr{E}_0$, it clearly holds for every smaller $\mathscr{E}'_0$: this defines for every $\mathscr{C}_p$ a set of p distinct domains which are continuously foliated by invariant ellipses: such a multiple domain will be called the "stable basin" of $\mathscr{C}_p$. This basin is stable in the sense that small perturbations of T imply small deformations of ellipses.

Note 2. The characteristic sequence $\{\mathbf{n}_k\}$ is the same for all points (also nonperiodic) in the stable basin of $\mathscr{C}_p$. Being associated to stable basins and not only to their central periodic orbits, these sequences are therefore easily computable.

Note 3. Since T is unitary, $\tilde{T}$ preserves *locally* (i.e., for sets contained in a single basin) the measure $dx\,dy$, even if the dynamical system is globally dissipative.

Note 4. Condition (3) is violated when, for a certain s, the point $\tilde{T}^s\tilde{\mathbf{z}}$ overpasses a side of the unit square: then, a change in the sequence is required, i.e., up to the step $s-1$ the evolution represents a transient before the definitive settlement in a basin (an example is given in Fig. 3). This means that ellipses are attractors in a larger set, the total basin of the periodic orbit. As we shall see below, in certain conditions elliptic domains may deform into polygonal ones.

Let now $\tilde{T}$ depend continuously on a parameter θ, as in (1). Then every orbit $\mathscr{C}_p$ and its stable basin will also depend continuously on θ, at least for an interval of values of this parameter. We expect, for instance, that a basin will exist as long as one of the centers reaches the border of $\mathscr{T}^2$. Moreover, let $\theta+\varepsilon$ be such that the corresponding rotation ϕ_ε is irrational with respect to 2π: then orbits fill densely $\tilde{T}$-invariant ellipses in the basin. If, for $\varepsilon \to 0$, $\phi = \lim \phi_\varepsilon$ is rational (i.e., $\phi/2\pi = m/q$), the slightly modified basin is still foliated, but the ellipses are no longer densely filled because orbits on them are periodic, too. So, the foliated structure of the basin is preserved for small, continuous variations of θ, while the behavior on the ellipses is not.

Such qualitative considerations will now be made more precise.

Let $\mathscr{C}_p = \mathscr{C}_p(\theta)$ be a p-periodic orbit with a characteristic sequence $\{\mathbf{n}_k\}$. Then

$$T\tilde{\mathbf{z}}_k - \mathbf{n}_k = \tilde{\mathbf{z}}_{k+1}$$

$$T[T\tilde{\mathbf{z}}_k - \mathbf{n}_k] - \mathbf{n}_{k+1} = T^2\tilde{\mathbf{z}}_k - T\mathbf{n}_k - \mathbf{n}_{k+1} = \tilde{\mathbf{z}}_{k+2}$$

and iterating p times

$$T^p\tilde{\mathbf{z}}_k - \sum_{j=1}^{p} T^{p-j}\mathbf{n}_{k+j-1} = \tilde{\mathbf{z}}_{k+p} \equiv \tilde{\mathbf{z}}_k$$

This may be written

$$(T^p - I)\tilde{\mathbf{z}}_k = \sum_{j=1}^{p} T^{p-j}\mathbf{n}_{k+j-1}$$

and, since every point may be assumed as the "first" one, putting $k=1$

$$(T^p - I)\tilde{\mathbf{z}}_1 = \sum_{j=1}^{p} T^{p-j}\mathbf{n}_j \tag{4}$$

If $R_p = R_p(\theta) = (T^p - I)^{-1}$ exists, Eq. (4) is formally solved by

$$\tilde{\mathbf{z}}_1 = R_p \sum_{j=1}^{p} T^{p-j}\mathbf{n}_j \tag{5}$$

Since the characteristic sequence may be computed (Note 2 to the Lemma), formula (5) gives the coordinates of the first periodic point in $\mathscr{C}_p$ [the others may be obtained by a cyclical permutation of the sequence in (5), or simply applying the automorphism to $\tilde{\mathbf{z}}_1$]. Using the Cauchy–Dunford formula,[6] the rhs of (5) assumes a definite and computable form:

$$\tilde{\mathbf{z}}_1 = \frac{2 \sin \phi \sin p\phi/2}{(1-\cos 2\phi)(1-\cos p\phi)} \times \sum_{j=1}^{p} \left\{ \cos\left[\left(\frac{p}{2}-j\right)\phi\right] T^{-1} - \cos\left[\left(\frac{p}{2}-j+1\right)\phi\right] I \right\} \mathbf{n}_j \tag{6}$$

Solution (6) of system (4), when R_p exists, is unique: hence two basins with the same characteristic sequence cannot coexist, and the number of basins with periodicity up to p is finite. Now, the existence of R_p depends both on p, the periodicity of the orbit, and ϕ. More precisely, when $\phi/2\pi = m/q$, the basin consists of periodic points with periodicity $\tilde{p} = qp/M(q, p)$, where $M(q, p)$ is the highest comon factor of its arguments. If, moreover, $p = rq$ (r entire), then $\tilde{p} = p$, $T^p = I$, and all points in the stable basin are p-periodic.[3,5]

But when $T^p = I$, R_p does not exist, and Eq. (4) cannot be solved through (5) or (6). We want to show that, even in this case, it is possible to obtain a solution $\tilde{\mathbf{z}}_1$ by a limit procedure on (6).

Theorem. The limit of (6) for $\phi \to 2\pi m/p$ exists and it is finite, even when the periodicity of the basin is equal to or a multiple of the periodicity p of T.

Proof. For $\phi/2\pi = m/p$, noninvertibility of $T^p - I$ corresponds to a singularity in the coefficient of (6): indeed, the double zero of $(1-\cos p\phi)$ in the denominator is not compensated by the simple zero of $\sin(p\phi/2)$. But since in general

$$\tilde{T}^p \tilde{\mathbf{z}} = T^p \tilde{\mathbf{z}} - \sum_{j=1}^{p} T^{p-j} \mathbf{n}_j$$

and since, in this case, $\tilde{T}^p \tilde{\mathbf{z}} = \tilde{\mathbf{z}}$ for every $\tilde{\mathbf{z}}$ and $T^p = I$, it follows that

$$\sum_{j=1}^{p} T^{p-j} \mathbf{n}_j = 0 \tag{7}$$

Therefore also the sum in the rhs of (5) [or (6)] vanishes, and the singularity may be removed. By straightforward calculation on (6) [e.g., via de l'Hôpital's rule) we obtain indeed

$$\tilde{\mathbf{z}}_1 = \frac{2\sin\phi}{p(1-\cos 2\phi)} \sum_{j=1}^{p} \left\{ -\left(\frac{p}{2}-j\right) \sin\left[\left(\frac{p}{2}-j\right)\phi\right] T^{-1} + \cos\left[\left(\frac{p}{2}-j\right)\phi\right] \frac{\partial}{\partial\phi} T^{-1} + \left(\frac{p}{2}-j+1\right) \sin\left[\left(\frac{p}{2}-j+1\right)\phi\right] I \right\} \mathbf{n}_j \tag{8}$$

Expression (8) represents the limit we looked for.

Note. A further singularity appears in (6) for $1-\cos 2\phi = 0$, i.e., $p=2$, $\phi = m\pi$, or $\lambda_{\pm} = \pm 1$: but these values separate elliptic from hyperbolic matrices, and transition to hyperbolicity destroys the basins with the appearance of an "attractor" whose features (e.g., the type of fractality) have to be investigated case by case.

As a corollary, in the same hypothesis the shape of a stable basin changes discontinuously from ellipses into polygons. We may give a short hint on their generation: let $\tilde{\mathbf{z}}$ be a generic point in the stable basin (say, in the neighborhood of $\tilde{\mathbf{z}}_1$). Then, $\tilde{T}^k \tilde{\mathbf{z}} \in \mathscr{T}^2$ for $k=1,\dots,p$ may be written

$$\begin{cases} \tilde{\mathbf{z}} \in \mathscr{T}^2 \\ \tilde{T}\tilde{\mathbf{z}} \equiv T\tilde{\mathbf{z}} - \mathbf{n}_1 \in \mathscr{T}^2 \\ \tilde{T}^2\tilde{\mathbf{z}} \equiv T^2\tilde{\mathbf{z}} - T\mathbf{n}_1 - \mathbf{n}_2 \in \mathscr{T}^2 \\ \vdots \\ \tilde{T}^{p-1}\tilde{\mathbf{z}} \equiv T^{p-1}\tilde{\mathbf{z}} - \sum_{j=1}^{p-1} T^{p-j}\mathbf{n}_j \in \mathscr{T}^2 \end{cases} \tag{9}$$

The successive condition on $\tilde{T}^p\tilde{\mathbf{z}}$ would replicate the first one: indeed, $T^p = I$ and the sum vanishes because of (7). Relations (9) may be made explicit: for instance, the second one gives

$$\begin{cases} 0 \leqslant t_1 x + t_2 y - n_1^{(x)} < 1 \\ 0 \leqslant t_3 x + t_4 y - n_1^{(y)} < 1 \end{cases} \tag{10}$$

where $n_1^{(x)}$ and $n_1^{(y)}$ denote the components of $\mathbf{n}_1$, and

$$T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$$

Because of (9), there are p pair of inequalities like (10). Each of them individuate a strip between parallel lines, and the intersection of p such strips generates a polygon around $\tilde{\mathbf{z}}_1$. Since the choice of the first neighbor is arbitrary, there are p such polygons.

The same arguments show that polygons also arise for $\phi/2\pi = m/q$ and $\tilde{p} \neq p$. For irrational ϕ, on the contrary, there is an infinity of distinct relations (9): in such a case the intersection of infinite strips generates indeed ellipses around central periodic points.

The components x_k and y_k of $\tilde{\mathbf{z}}_k$ obey the conditions:

$$0 \leqslant x_k < 1, \qquad 0 \leqslant y_k < 1 \tag{11}$$

Inequalities (11) apply to the lhs of (6), providing conditions on the angle ϕ (implicitly on θ) for the existence of the p-periodic orbit with its basin. For every k one obtains indeed a pair $\theta_{k,\min}$, $\theta_{k,\max}$ of bounds, and finally a pair

$$\begin{cases} \theta_{\min} = \max\limits_k \{\theta_{k,\min}\} \\ \theta_{\max} = \min\limits_k \{\theta_{k,\max}\} \end{cases} \tag{12}$$

In principle, values (12) of θ are well defined and solve the problem of finding the existence range for a certain $\mathscr{C}_p$ with its basin. But the direct computation of bounds (12) from (6) appears to be cumbersome. Moreover, at every θ, there coexist several basins, and their splitting is not reducible in general to a simple rule: the number, multiplicity, and range of existence of the new basins depend indeed in a nontrivial way on both T and θ. This fact makes a Feigenbaum-like scheme unseccessful.

Nevertheless, solutions (6) may be easily plotted. It is simple to follow graphically their evolution vs. θ, as shown, for instance, in Figs. 4 and 5.

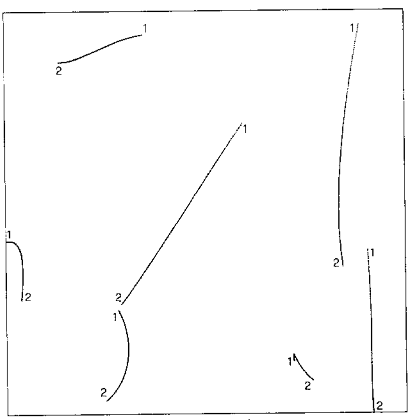

Fig. 4. Evolution of central points for the period-7 basin of Fig. 2. Position 1 corresponds to $\theta = 1.2650$, position 2 to $\theta = 1.43545$.

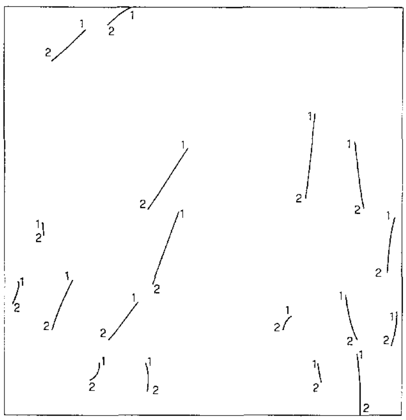

Fig. 5. Evolution of central points for the period-18 basin of Fig. 2. Position 1 corresponds to $\theta = 1.3058$, position 2 to $\theta = 1.3665$.

This provides good approximate estimates for bounds (12). Moreover, the mechanism determining the features of motion when, for a θ_0, there is an elliptic-hyperbolic transition [as in example (1)] may now be clarified. Since the ϕ rotation STS^{-1} on circles in $\mathbf{R}^2$ is uniform, in $\mathcal{T}^2$ the local density grows with the local curvature of orbits. In other words, dense orbits accumulate at the acute extremities of ellipses. When $\theta \to \theta_0$, eccentricity diverges because ellipses degenerate into segments and basins are squeezed into parallel lines (Figs. 6 and 7). Being tangent to one another, basins no longer exist and periodic or quasiperiodic motion is

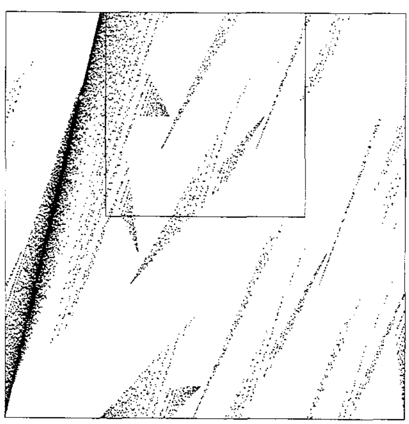

Fig. 6. A single orbit on the strange attractor at $\theta_0 = 0.8410686706$.

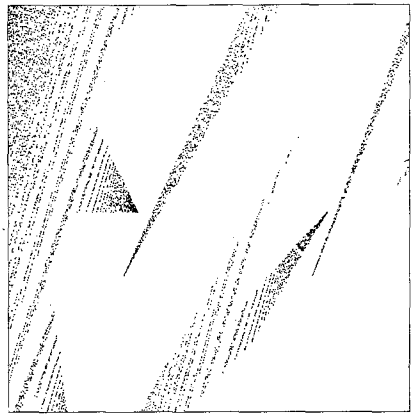

Fig. 7. An enlarged part of Fig. 6, displaying the fine structure of the attractor.

destroyed; but a sort of "memory" of the local time of sojourn on stretched ellipses may persist in the (possibly highly irregular) behavior on the attracting set. The final features of the attractor (its strangeness, the stochastic measure on it) inherit therefore those of collapsing basins, whose dynamics we have studied above.

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Communicated by J. L. Lebowitz